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CONDITIONAL GRAPH COMPLETIONS

by

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If $G=(V,E)$ is a simple graph of order p and size q, and if P is a property held by G, we say that G is *P-completable* if there is an ordering $e_1, e_2, \ldots, e_{\binom{p}{2}-q}$ of the edges of $K_p - G$ such that $G_k = \left(V, E + \bigcup_{i=1}^{k} e_i\right)$ has property P for each $k = 1, 2, \ldots, \binom{p}{2} - q$. The sequence $\{G_k\}$ is called a *P-completion sequence*. If all graphs with property P are P-completable, we say that P is a *completable* property and that the the class Π of graphs with property P is a *completion* class. Of interest are *conditional completion classes*, i.e., classes for which not all orderings lead to completion sequences. We show that several familiar classes of graphs are conditional completion classes.

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CONDITIONAL GRAPH COMPLETIONS

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Abstract. If $G = (V, E)$ is a simple graph of order p and size q, and if P is a property held by G, we say that G is *P-completable* if there is an ordering $e_1, e_2, \ldots, e_{\binom{p}{2}-q}$ of the edges of $K_p - G$ such that $G_k = \left(V, E + \bigcup_{i=1}^{k} e_i\right)$ has property P for each $k = 1, 2, \ldots, \binom{p}{2} - q$. The sequence $\{G_k\}$ is called a *P-completion sequence*. If all graphs with property P are P-completable, we say that P is a *completable* property and that the the class Π of graphs with property P is a *completion* class. Of interest are *conditional completion classes*, i.e., classes for which not all orderings lead to completion sequences. We show that several familiar classes of graphs are conditional completion classes.
Keywords: Chordal graphs, perfect graphs, matrix completions.

1. Preliminaries. Unless otherwise specified, all graphs are assumed to be simple. If $G = (V, E)$ is a graph, and if $V' \subseteq V$, then we denote by $\langle V' \rangle$ the subgraph of G induced by V'. Given a graph $G = (V, E)$, we denote by $\overline{G}$ the *complement* of G, i.e., $\overline{G} = (V, \overline{E})$, where $\overline{E} = \{xy | x, y \in V, xy \notin E\}$. We use $X + Y$ to denote the union of disjoint sets X, Y. If $G = (V, E)$, $F \subseteq E(\overline{G})$, and $f \in F$, then it is convenient to let $G + f$ denote the graph $G' = (V, E + \{f\})$. Similarly, if $e \in E$, we let $G - e$ denote the graph $G' = (V, E \backslash \{e\})$. A graph $G = (V, E)$ is *chordal* if G contains no induced k-cycle for $k \geq 4$. The *neighborhood* $N(v)$ of a vertex $v \in V$ is defined by $N(v) = \{x \in V | vx \in E\}$; the *closed neighborhood* of v is given by $N[v] = N(v) + \{v\}$. A vertex v in G is *simplicial* if $\langle N(v) \rangle$ is a *clique*, i.e., a collection of vertices that induces a complete subgraph. A clique need not be maximal. If $V' \subseteq V$, $V' \neq \emptyset$, and $\langle V' \rangle = (V', \emptyset)$, then V' is an *independent set*. We denote by $\omega(G)$ the order of a largest clique in G, and by $\chi(G)$ the chromatic number of G. For terminology not defined here, see Bondy [1].

It is a well-known fact, first reported by Dirac [3], that every chordal graph possesses a simplicial vertex. This phenomenon is the basis for an efficient recognition algorithm for chordal graphs due to Fulkerson and Gross [5]. They show that every chordal graph G has a *perfect elimination ordering*, a labeling of the vertices as $v_1, v_2, \ldots, v_n$ such that for each $1 \leq i \leq n$, v_i is simplicial in $\langle v_i, v_{i+1}, \ldots, v_n \rangle$.

Given a family $\mathcal{F} = \{F_1, F_2, \ldots, F_k\}$ of subsets of some universal set, we may construct the *intersection graph* G of $\mathcal{F}$ by letting the vertices of G be the elements of $\mathcal{F}$ and including the edge F_iF_j if and only if $F_i \bigcap F_j \neq \emptyset$. An *interval graph* is a graph G that can be represented as the intersection graph of a set of intervals on the real line. If this can be accomplished using intervals of constant length, G is said to be *unit interval*. For details on constructing (unit) interval representations of (unit) interval graphs, see Roberts [11, 12]. Another characterization of interval graphs is due to Fulkerson and Gross [5]. They show that G is an interval graph if and only if the maximal cliques of G can be ordered in such a way that, for any vertex $x \in V$, the maximal cliques containing x occur consecutively. Yet another characterization is due to Lekkerkerker and Boland [10], who use a forbidden subgraph approach. The details are not needed here, but a useful corollary to their result is that any chordal graph on five or fewer vertices is an interval graph. We may similarly apply a result of Roberts [11] to deduce that any graph on three or fewer vertices is unit interval.

A graph G is said to be *perfect* if $\chi(G') = \omega(G')$ for all induced subgraphs $G' \subseteq G$. Many well-known classes of graphs fall into the class of perfect graphs, including the chordal graphs and the bipartite graphs. Certain classes of perfect graphs arise in applications, and many classes of perfect graphs have desirable algorithmic properties. For a compilation of much that is known about perfect graphs, see Golumbic [7].

The notion of a *hereditary property* is well understood in graph theory: P is said to be a hereditary property if, whenever a graph G has property P, all (vertex-) induced subgraphs of G have property P. For example, the property of being chordal, the property of being acyclic, and the property of being interval all are hereditary properties. If P is a hereditary property, and G a graph with property P, it is typically quite easy to show that the inheritance does not extend to *edge*-induced subgraphs of G. An obvious exception is the property of being acyclic, but if G is, say, a chordal graph containing a (noninduced) cycle of length at least four, then we may readily find an edge-induced subgraph of G that is not chordal.

Surprisingly, though, if G is chordal and incomplete, then we may always *add* an edge to G is such a way that the resulting supergraph is chordal. While implicit in the work of Rose, Tarjan, and Lueker [14], this phenomenon first appears in the work of Johnson et al. [8] in the context of the partial positive definite matrix completion problem. We may generalize this phenomenon in the following way. Let $G = (V, E)$ be a graph of order p and size q, and suppose that G has some property P. We say that G is *P-completable* if the edges of $\overline{E}$ can be added serially in such a way that each supergraph in the resulting sequence has property P; the associated ordering $e_1, e_2, \ldots, e_k$, where $k = \binom{p}{2} - q$, is a *P-completion sequence* or, when no ambiguity is likely to arise, simply a *completion sequence*. For convenience, we will sometimes refer to the resulting sequence of graphs as the completion sequence.

If all graphs with property P are P-completable, we say that P is a *completable property* and that the class Π of graphs with property P is a *completion class*. Some properties are such that *any* ordering of the edges missing from a representative G will induce a P-completion sequence. For example, the class of all connected graphs on p vertices for fixed p is a trivial completion class. We are interested in nontrivial cases. In general, we shall refer to such classes as *conditional* completion classes; since only conditional completion classes will be considered here, we shall omit the adjective.

2. Chordal Completion Classes. The proof [8] that the class of chordal graphs is a completion class depends upon the existence of perfect elimination orderings. Specifically, the proof rests on a result of Rose et al. [14], which is stated without proof as Lemma 2.1.

LEMMA 2.1 (ROSE, TARJAN, & LUEKER, 1976). *Let $G = (V, E)$ be chordal, and suppose that $G' = (V, E + F)$ is also chordal for some nonempty F satisfying $E \bigcap F = \emptyset$. Then there exists some $f \in F$ such that $G' - f$ is chordal.* Letting $G = (V, E)$ be a chordal graph of order p and size q, with $k = \binom{p}{2} - q$, we may apply this result to find a sequence $e_1, e_2, \ldots, e_k$ of edges such that each of the graphs $G_0, G_1, \ldots, G_k = G$ is chordal, where $G_0 \cong K_p$ and $G_i = G_{i-1} - e_i$. But now we see that $e_k, e_{k-1}, \ldots, e_1$ is a completion sequence for G, as observed by Johnson et al. [8]. We use a more direct approach to show that the class of interval graphs is a completion class. Note that this is done without exploiting perfect elimination orderings. While interval graphs are chordal, and so possess such orderings, it is perhaps clearer from an intuitive point of view to exploit the defining property of interval graphs.

THEOREM 2.2. *Let G be an interval graph. If G is not complete, then G allows an interval completion sequence.*

Proof: Suppose that $G = (V, E)$ is an incomplete interval graph with vertex set $V = \{v_1, v_2, \ldots, v_p\}$. Since all chordal graphs on fewer than six vertices are interval, we may assume that $p \geq 6$. If G is empty, we may insert any edge with no danger of losing the interval property. Otherwise let $C_1, C_2, \ldots, C_m$ be a consecutive ordering of the maximal cliques of G. Without loss of generality we may assume that the vertices in V are labeled in such a way that $v_i \in C_j \backslash C_k$, $v_l \in C_k$, and $j < k$ together imply $i < l$. That such a labeling can be found follows from the Fulkerson and Gross characterization of interval graphs [5]. It suffices to show that an edge $f \notin E$ can be found such that $G^* = G + f$ is interval, the result then following inductively. Let $j = \min\{k | v_i v_k \notin E$ for at least one $i < k\}$, and let $i = \min\{k | v_k v_j \notin E\}$. Let $f = v_i v_j$. Then $G^* = G + f$ is interval. To see this, consider an interval representation $\mathcal{I} = \{I_1, I_2, \ldots, I_k\}$ of G. Let $I_j = [l_j, r_j]$, and suppose that $i < j$ implies that $l_i < l_j$. Then j, determined above, is the index of the leftmost (as determined by its left endpoint) interval that fails to overlap at least one interval that lies to its left, while i is the index of the leftmost interval that fails to overlap I_j. Adding $f = v_i v_j$ to G is then equivalent to "stretching" I_i by shifting r_i to the right, leaving l_i fixed, until I_i overlaps I_j. By our choice of i and j, the result is $\mathcal{I}'$, a collection of intervals whose intersection graph is $G + f = G^*$. □

The procedure described above for interval graphs obviously does not work for unit interval graphs, since the intervals in that case must be of constant length. However, a modification of that procedure can work. Rather than stretch an interval, we simply shift it along the real line until it is *almost* identified with its closest neighbor, as described in the proof of the following theorem.

THEOREM 2.3. *Let G be a unit interval graph. If G is not complete, then G allows a unit interval completion sequence.*

Proof: Let $G = (V, E)$ be an incomplete unit interval graph, and let $\mathcal{I} = \{I_1, \ldots, I_p\}$ be an interval representation of G as before, but with the additional property that each interval I_k is now of unit length.

As before, if l_i is the left endpoint of I_i then we have $l_1 < l_2 < \cdots < l_p$. Choosing some suitably small quantity ε, shift I_1 to the right, if possible, until $l_2 - l_1 = \varepsilon$. In so doing, $N(v_1)$ and $N(v_2)$ merge, one vertex at a time, yet the resulting graph is still unit interval. Now "glue" I_1 and I_2 together, and slide the resulting object to the right until $l_3 - l_2 = \varepsilon$. Continuing in this fashion, alternately shifting and gluing,

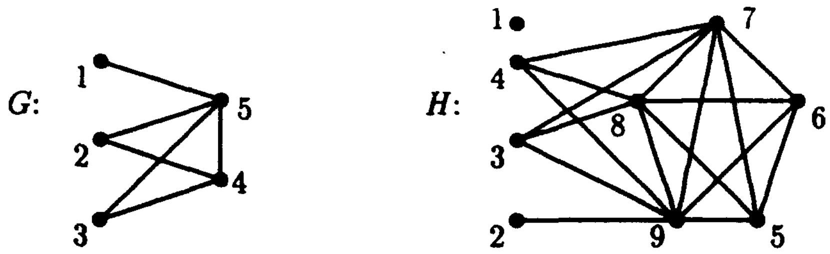

FIG. 1. *Two split graphs. Both are threshold graphs, as well.*

we ultimately arrive at a situation where $\bigcap_{k=1}^{p} I_k \neq \emptyset$, and the process is complete.□

A *split graph* $G = (V, E)$ is a connected graph that allows a partition $V = \{K, I\}$ such that K is a clique and I is an independent set. See figure 1 for examples of split graphs. We show that the split graphs constitute a completion class. Given an arbitrary split graph $G = (K + I, E)$, with $|K| \geq 2$, $|I| \geq 2$, it is clear that if $u, v \in I$ then $G + uv$ is not necessarily a split graph. The following simple result shows that the split graphs are a completion class.

THEOREM 2.4. *Let G be a split graph. If G is not complete, then G allows a split completion sequence.*

Proof: Let $G = (V, E)$ be an incomplete split graph, and let $\{K, I\}$ be a partition of V such that K is a clique and I is an independent set. If $K = \emptyset$ then we may insert any edge, so assume that G is nonempty. It suffices to show that we may find at least one edge xy such that $G + xy$ is a split graph, since the result then follows inductively. First suppose that some edge xy is missing, where $x \in K$ and $y \in I$. Then $G + xy$ is a split graph with partition $V = \{K, I\}$. On the other hand, if each vertex in K is adjacent to every vertex in I, then since G is not complete we know that $|I| \geq 2$. Let $x, y \in I$. Then $G + xy$ is a split graph with corresponding partition $V = \{K + \{x, y\}, I - \{x, y\}\}$. □

3. Other Completion Classes. Each of the classes of graphs that has so far been shown to be a completion class has been a chordal class. We now obtain similiar results for several classes that contain nonchordal graphs.

A *circular-arc* graph is one that can be represented as the intersection graph of arcs on a circle. It is easy to see that every interval graph is also a circular-arc graph, and that the converse fails. In fact, while interval graphs are chordal, hence perfect, circular-arc graphs in general possess neither property. Nevertheless, by selecting a starting arc arbitrarily and by then modifying the procedures described above for interval graphs to work, say, clockwise instead of left-to-right, we obtain the following corollary.

COROLLARY 3.1. *Let $G = (V, E)$ be a (proper) circular-arc graph. If G is not complete, then G allows a (proper) circular-arc completion sequence.*

A *comparability graph* is a graph $G = (V, E)$ that allows a transitive orientation of its edges, i.e., an orientation F with the property that for any vertices $x, y, z \in V$, if $(x, y), (y, z) \in F$ then $(x, z) \in F$. It is well known that such an orientation is acyclic, and that a linear ordering $v_0, v_1, \ldots, v_{n-1}$ of V may be found that is compatible with

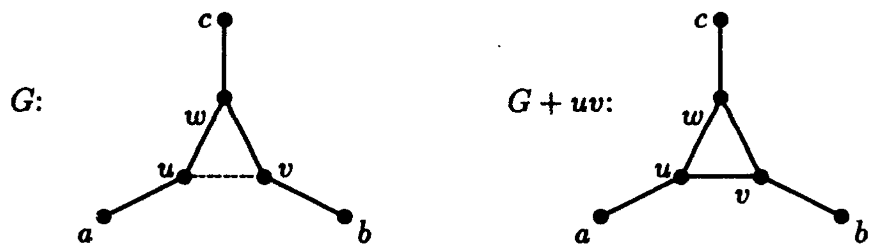

FIG. 2. *G is a comparibility graph, G + uv is not.*

F, which is to say that for any $i \neq j$, if $(v_i, v_j) \in F$, then $i < j$. Such an ordering is called a *topological ordering* of V. See, for example, Golumbic [7] or Roberts [13]. This is a special case of the *level assignment* described by Harary, Norman, and Cartwright [9] and applied by them to obtain the (simultaneous) completion of an acyclic digraph. Figure 2 shows a comparability graph G. If we add to G the missing edge uv, the resulting graph is easily seen not to be a comparability graph. We now show that the comparability graphs constitute a completion class.

THEOREM 3.2. *Let G be a comparability graph. If G is not complete, then G allows a comparability completion sequence.*

Proof: Let $G = (V, E)$ be a comparability graph of order n, not complete. Let F be a transitive orientation of E. Let $v_0, v_1, \ldots, v_n$ be a topological ordering of V that is compatible with F. Let $j = \max\{k | \text{in}(v_k) < k\}$, and let $i = \min\{l | v_i v_l \notin E\}$. We must show that $G' = (V, E + v_i v_j)$ is a comparability graph. It suffices to show that $F + (v_i, v_j)$ is a transitive orientation of the edges of G'. Suppose that transitivity is violated, so that for some k, l and m we have arcs (v_k, v_l) and (v_l, v_m) but that (v_k, v_m) is missing. Evidently either $k = i$ and $l = j$ or $l = i$ and $m = j$, since F is a transitive orientation of E. Suppose that $k = i$ and $l = j$. Since our vertex labeling is compatible with F, we know that $i > j > m$. Moreover, by our choice of j, we know that $v_i v_m \in E$. Since $(v_i, v_m) \notin F$, then it must be that $(v_m, v_i) \in F$, but then $\langle v_i, v_j, v_m \rangle$ is a directed cycle in F, a contradiction. So it must be that $l = i$ and $m = j$. By our labeling scheme, we know that $k > i > j$. By our choice of i, $v_k v_j \in E$. Since $(v_k, v_j) \notin F$, then $(v_j, v_k) \in F$, and we again find a directed cycle in F. Since this cannot occur, we conclude that $F + (v_i, v_j)$ is a transitive orientation of $G + v_i v_j$, showing that G' is a comparability graph. □

Given a permutation $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ of $\{1, 2, \ldots, n\}$, we denote $(\alpha^{-1})_i$, the position of i in α, by α_i^{-1}. An *inversion* is a pair $\{i, j\} \in \{1, 2, \ldots, n\}$ with the property that i is smaller than j but $\alpha_i^{-1} > \alpha_j^{-1}$. We may construct a graph $G(\alpha) = (V, E)$ with $V = \{1, 2, \ldots, n\}$ and $E = \{ij | \{i, j\}$ is an inversion in $\alpha\}$. Such a graph is called a *permutation graph*. If we let $\pi = (1, 2, \ldots, n)$ then $G(\pi) = I_n$, while if we let $\beta = (n, (n-1), \ldots, 1)$ then $G(\beta) = K_n$. It is easy to show that a permutation graph is transitively orientable, so we obtain as an immediate corollary to the preceding theorem the fact that permutation graphs have comparability completion sequences. However, not all such sequences will maintain the permutation property. The correspondence between permutation graphs of order n and permutations of $\{1, 2, \ldots, n\}$ suggests an algorithmic proof. Such a construction exists, and

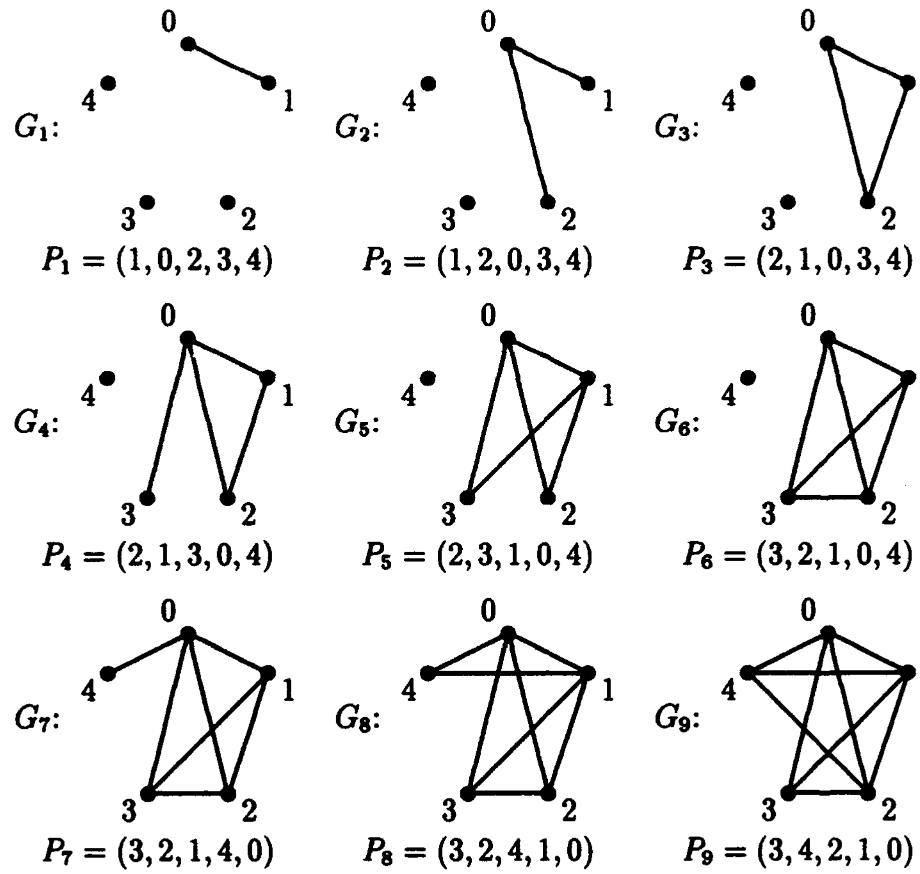

FIG. 3. *Permutation completion sequence.*

establishes the following result.

THEOREM 3.3. *Let G be a permutation graph. If G is not complete, then G allows a permutation completion sequence.*

Proof: Let $G = G_0$ be a permutation graph of order p, not complete, and let $\pi = \pi_0$ be the permutation of $\{1, 2, \ldots, p\}$ corresponding to G. If we apply the standard bubble sort algorithm to reorder π in descending order, we obtain a sequence of permutations $\pi_0, \pi_1, \ldots, \pi_k$, where $\pi_k = (p, p-1, \ldots, 1)$, and for any $0 \leq i \leq k-1$, π_{i+1} is obtained from π_i by an interchange, i.e., π_{i+1} differs from π_i by exactly one inversion. It is easy to see that we obtain a corresponding sequence of permutation graphs $G_0, G_1, \ldots, G_k$ with the property that $G_k = K_p$ and, for each $0 \leq i \leq k$, G_{i+1} is obtained from G_i by the insertion of exactly one edge. Thus $G_0, G_1, \ldots, G_k$ is a permutation completion sequence. □

For an illustration of the operation of the algorithm, we give as input the graph $G = G_0 \cong I_5$. We denote by P_k the permutation corresponding to G_k. The corresponding permutation of $\{0,1,2,3,4\}$ is $P_0 = (0,1,2,3,4)$. The subsequent permutations and their associated graphs are shown in Figure 3. Not shown are G_0, P_0, $G_{10} \cong K_5$, and P_{10}, the reversal of P_0.

An interesting subclass of the permutation graphs is the class of *threshold graphs*, introduced by Chvátal and Hammer [2] and discussed in some detail in Golumbic [7]. Most useful in the current context is the characterization of a threshold graph in terms of a degree partition of its vertex set. Adopting the notation used by Golumbic, we let $0 < \delta_1 < \cdots < \delta_m < |V|$ denote the degrees of the vertices of G. Let $\delta_0 = 0$. For each $i = 0, 1, \ldots, m$, let $D_i = \{x \in V | \deg(x) = \delta_i\}$. By definition of the δ_i, all except possibly D_0 must be nonempty, and the sets D_i induce a partition of V. The characterization due to Chvátal and Hammer [2] shows that a graph G with degree partition $V = D_0 + D_1 + \cdots + D_m$ is a threshold graph if and only if for every distinct pair $x \in D_i$, $y \in D_j$, $x \neq y$, $xy \in E$ if and only if $i + j > m$. Thus for $1 \leq i \leq m$, $x \in D_i$ implies that $N[x] = \bigcup_{j=1}^{i} D_{m-j+1}$, and for $1 \leq i \leq \lfloor m/2 \rfloor$, $\bigcup_i D_i$ is an independent set, while for $\lceil m/2 \rceil \leq i \leq m$, $\bigcup_i D_i$ is a clique. See figure 1 for examples of threshold graphs G and H. In G, we have $\delta_i = i$ for $i = 1, 2, 3, 4$, and the degree partition of $V(G)$ is given by $D_1 = \{1\}, D_2 = \{2, 3\}, D_3 = \{4\}$, and $D_4 = \{5\}$. In H, we have $\delta_1 = 1, \delta_2 = 3, \delta_3 = 4, \delta_4 = 6$, and $\delta_5 = 7$. The sets in the degree partition of $V(H)$ are $D_0 = \{1\}, D_1 = \{2\}, D_2 = \{3, 4\}, D_3 = \{5, 6\}, D_4 = \{7, 8\}$, and $D_5 = \{9\}$.

That every threshold graph has split, comparability, and permutation completion sequence follows from theorems 2.4, 3.2, and 3.3, respectively. Showing that every threshold graph has a threshold completion sequence requires somewhat more effort in devising a successful strategy for the selection of entering edges. We exploit the degree partition described in the previous paragraph.

THEOREM 3.4. *Let G be a threshold graph. If G is not complete, then G allows a threshold completion sequence.*

Proof: Suppose that $G = (V, E)$ is incomplete and that $V = \{D_0, D_1, \ldots, D_m\}$ is a degree partition of V as described above. Assume that G is nonempty, since we can then insert any edge. It suffices to show that we may find an edge $e \in \overline{E}$ with the property that $G + e$ is a threshold graph, the result then following inductively. We have several cases to consider.

First suppose that $D_0 \neq \emptyset$. Let $x \in D_0$ and $y \in D_m$. Consider $G^* = G + xy$. If $\delta_1 = 1$ in G, then $D_m = \{y\}$, so when xy is inserted the only changes are that x leaves D_0 to enter D_1 and δ_m is incremented. The number m of sets in the degree partition of the nonisolated vertices does not change. If, on the other hand, $\delta_1 > 1$ in G, then D_m contains at least one vertex $z \neq y$. When xy is inserted, the degree of x rises to $1 < \delta_1$, so a new singleton set $D_x = \{x\}$ is induced. Similarly the degree of y rises to $\delta_m + 1$, and so a singleton $D_y = \{y\}$ is created. To verify that G is a threshold graph, it is necessary to relabel the sets in the degree partition as $D_1^*, D_2^*, \ldots, D_{m^*}^*$, taking into account that we now have vertices of $m^* = m + 2$ different degrees. Thus $D_1^* = D_x$, $D_2^* = D_1 - \{x\}$, $D_{m^*-1}^* = D_m - \{y\}$, $D_{m^*}^* = D_y$, and for each $3 \leq k \leq m - 1$ we have $D_k^* = D_{k-1}$. It is now straightforward to verify that for any vertices $s \in D_i^*$, $t \in D_j^*$, $s \neq t$, the edge $st \in E + xy$ if and only if $i + j > m^*$, and that G' is therefore a threshold graph.

Now suppose that $D_0 = \emptyset$. If $m = 2$, then for any $x, y \in D_1$ we may insert the edge xy, whereupon either we have the complete graph or we have a graph G^* with degree partition $D_1^* = D_1 - \{x, y\}$, $D_2^* = \{x, y\}$, and $D_3^* = D_2$. Suppose, then, that $m > 2$. Let $x \in D_1$, $y \in D_{m-1}$. Consider $G^* = G + xy$. We now have two possiblities to consider.

Case 1: In G, $D_1 = \{x\}$. Then $\delta_m = 1 + \delta_{m-1}$. If $D_{m-1} = \{y\}$, i.e., $\delta_2 = 1 + \delta_1$, then when the edge xy is inserted the sets D_1, D_2 merge, as do the sets D_m, D_{m-1}. We now have sets $D_1^* = D_1 + D_2$, $D_{m^*}^* = D_m + D_{m-1}$, $m^* = m - 2$, and $D_i^* = D_{i+1}$ for each $i = 2, \ldots, m-3$. If $|D_{m-1}| > 1$, then $\delta_2 > 1 + \delta_1$. Vertex y enters D_m, leaving at least one vertex behind in D_{m-1}, and δ_1 rises, but the number, m, of sets is unchanged.

Case 2: In G, $|D_1| > 1$. Thus $\delta_m > 1 + \delta_{m-1}$. If $D_{m-1} = \{y\}$, i.e., $\delta_2 = 1 + \delta_1$, then when the edge xy is inserted x is absorbed into D_2, D_1 persists (without x), δ_{m-1} is incremented, and D_{m-1} is preserved. If $|D_{m-1}| > 1$, then $\delta_2 > 1 + \delta_1$. When xy is inserted, a new set $D_x = \{x\}$ splits off of D_1 and a new set $D_y = \{y\}$ splits off of D_{m-1}. We now have $D_1^* = D_1 - \{x\}, D_2^* = \{x\}, D_{m^*}^* = D_m, D_{m^*-1}^* = \{y\}$, $m^* = m + 2$, and $D_i^* = D_{i-1}$ for each $i = 3, \ldots, m-1 = m^* - 3$.

In both cases 1 and 2, it is straightforward to show that the inequalities governing adjacencies remain satisfied in $G + xy$, which is therefore a threshold graph. □

We may view the completion of threshold graphs from another point of view. Golumbic [6, 7] shows that the threshold graphs of order p can be placed in one-to-one correspondence with a subset of the permutations of $\{1, \ldots, p\}$ in the following way. If $\{a\} = a_1, a_2, \ldots, a_m$ and $\{b\} = b_1, b_2, \ldots, b_k$ are sequences, then the *shuffle product* $\{a\} \sqcup\!\sqcup \{b\}$ is the set of sequences of the form $(\alpha_1\beta_1 \ldots \alpha_t\beta_t)$, where the α_i and β_i are subsequences of $\{a\}$ and $\{b\}$, respectively. Golumbic's result shows that G is a threshold graph of order p in which k vertices are independent and the remaining $p - k$ constitute a clique if and only if G is the permutation graph $G(\pi)$ of some permutation $\pi \in (p, p-1, \ldots, k+1) \sqcup\!\sqcup (1, 2, \ldots, k)$. For example, the graph G in figure 1 is the permutation graph of $\pi_G = (5, 1, 4, 2, 3)$, and the graph H in the same figure is the permutation graph of $\pi_H = (1, 9, 2, 8, 7, 3, 4, 6, 5)$. If we label the vertices of G in ascending order by degree, and if $|D_0| = c$, then the first $c + 1$ elements of π are $1, 2, \ldots, c, p$. The tactic described in the proof above, of either adjoining an isolated vertex $x \in D_0$ to a vertex $y \in D_m$ or of adjoining a pendant vertex $x \in D_1$ to a vertex $y \in D_{m-1}$ can be described in terms of a sorting operation on the associated permutation π. Scanning π from left to right, we look for the first occurrence of a subsequence of the form xy, where $x \in \{1, 2, \ldots, k\}$ and $y \in \{p, p-1 \ldots, k+1\}$. These elements are in their natural order, and so the edge xy is missing from G. We interchange x and y. The result is still an element of the shuffle product $(p, p-1, \ldots, k+1) \sqcup\!\sqcup (1, 2, \ldots, k)$, and so by Golumbic's characterization we know that the graph $G^* = G + xy$ is a threshold graph. If, in attempting this operation, we find no such subsequence, then it must be that $\pi = (p, p-1, \ldots, k+1, 1, 2, \ldots, k)$. In terms of the set representation we have $m = 2$, $D_1 = \{1, 2, \ldots, k\}$, and $D_2 = \{k+1, k+2, \ldots, p\}$. We now view π as an element of the shuffle product $(p, p-1, \ldots, k) \sqcup\!\sqcup (1, 2, \ldots, k-1)$, interchange k and $k-1$, and continue as before, eventually obtaining $\pi^* = (p, p-1, \ldots, 1)$ and the associated graph K_p.

With the exception of the circular-arc graphs, each completion class discussed thus far has been a perfect class. It is natural to consider the possibility that all perfect classes are completion classes, but we may easily show that this is not the case. For example, consider the comparability graphs of *weak orders*, i.e., irreflexive relations satisfying antisymmetry and negative transitivity. (See Fishburn [4] for details.) While these graphs have comparability completion sequences, by Theorem 3.2, it is not possible for each of the graphs in the sequence to be the comparability graph of a weak order. It is straightforward to show that such graphs are precisely those containing no induced subgraph isomorphic to $K_2 \bigcup K_1$. Moreover, for any $n \geq 3$, I_n and K_n have this property, yet it is clearly impossible to construct a completion sequence that will lead from I_n to K_n.

4. Directions For Further Work. This is preliminary work, and a number of unanswered questions remain. The existence of chordal completion sequences was useful in solving a problem with implications in the lore of numerical analysis and pure matrix analysis. What of these other completion sequences? It will be interesting to investigate their value when recast in the language of combinatorial matrix theory.

The only interesting perfect graphs that have been identified as failing to constitute a completion class are the comparability graphs of weak orders. These are characterized by a forbidden subgraph that is *disconnected*. Do there exist hereditary classes whose forbidden subgraphs are connected yet that are not completion classes?

This work has potential application in the arena of competition graphs. It might be possible to devise strategies for edge (arc) inclusions in a graph G (digraph D) that either preserve the chromatic number of the competition graph, if that is feasible, or that guarantee that the new competition graph will be chordal, or interval, say, and therefore easily colored.

A variation on this theme might involve questions of the following form. Suppose that G has property P, and that for some $x, y \in V(G)$, $xy \notin E(G)$. What is the smallest supergraph of G containing xy that has property P?

In general, it might be interesting to explore the idea of performing sequential edge inclusions until an extremal, or critical, case is constructed. If the largest graph with p vertices and having property P can be described, and if G has P, is there a P-preserving sequence of edge inclusions that will enable us to construct the extremal case?

Some of these questions are close to resolution and will be discussed in a forthcoming paper. Also forthcoming will be a discussion of practical algorithms for the construction of certain completion sequences.

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